

# Lotka-Volterra with randomly fluctuating environments: a full description

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## Abstract

In this note, we study the long time behavior of Lotka-Volterra systems whose coefficients vary randomly. Benaïm and Lobry established that randomly switching between two environments that are both favorable to the same species may lead to four different regimes: almost sure extinction of one of the two species, random extinction of one species or the other and persistence of both species. Our purpose here is to provide a complete description of the model. In particular, we show that any couple of environments may lead to the four different behaviours of the stochastic process depending on the jump rates.

## 1 Introduction

For a given set of positive parameters  $\varepsilon = (a, b, c, d, \alpha, \beta)$ , consider the Lotka-Volterra differential system in  $\mathbb{R}_+^2$ , is given by

$$\begin{cases} x' = \alpha x(1 - ax - by) \\ y' = \beta y(1 - cx - dy) \\ (x_0, y_0) \in \mathbb{R}_+^2 \end{cases}$$

We denote by  $F_\varepsilon$  the associated vector field:  $(x', y') = F_\varepsilon(x, y)$ . Let us note already that when  $a < c$  and  $b < d$ , the point  $(1/a, 0)$  attracts any path starting in  $(0, +\infty)^2$ . We say that the environment is favorable to species  $x$ . Similarly, when  $a > c$  and  $b > d$ , the point  $(0, 1/d)$  attracts any path starting in  $(0, +\infty)^2$ . We say that the environment is favorable to species  $y$ . See [6] for a detailed presentation of the four generic configurations. The environment is said to be of

- Type 1: if  $a < c, b < d$  (favorable to species  $x$ )
- Type 2: if  $a > c, b > d$  (favorable to species  $y$ )
- Type 3: if  $a > c, b < d$  (persistence)
- Type 4: if  $a < c, b > d$  (extinction of species  $x$  or  $y$  depending on the starting point)

Consider two such systems  $\varepsilon_0 = (a_0, b_0, c_0, d_0, \alpha_0, \beta_0)$  and  $\varepsilon_1 = (a_1, b_1, c_1, d_1, \alpha_1, \beta_1)$  and introduce the random process  $\{(X_t, Y_t, I_t)\}$  on  $\mathbb{R} \times \mathbb{R} \times \{0, 1\}$  obtained by switching between these two deterministic dynamics, at rates  $\lambda_0, \lambda_1$ . More precisely, we consider the Markov process driven by the following generator

$$Lf(z, i) = F_i(z) \cdot \nabla_z f(z, i) + \lambda_i(f(z, 1 - i) - f(z, i)), \quad (z, i) \in \mathbb{R}^2 \times \{0, 1\}.$$

Equivalently,  $(I_t)_{t \geq 0}$  is a Markov process on  $\{0, 1\}$  with jump rate  $\lambda_0$  and  $\lambda_1$ , that is

$$\mathbb{P}(I_{t+s} = 1 - i | I_t = i, \mathcal{F}_t) = \lambda_i s + o(s),$$

where  $\mathcal{F}_t$  is the sigma field generated by  $\{I_u, u \leq t\}$ . Finally,  $(X_t, Y_t)$  is solution of

$$(X'_t, Y'_t) = F_{\varepsilon_{I_t}}(X_t, Y_t).$$

This process on  $\mathbb{R}^2 \times \{0, 1\}$  has already been studied in [3, 6]. It belongs to the class of the piecewise deterministic Markov processes introduced by Davis [4]. See also [5] for a recent review of the application areas of such processes. Let us introduce the invasion rates of species  $x$  and  $y$  defined in [3] as

$$\begin{aligned} \Lambda_y &= \int \beta_0(1 - c_0 x) \mu(dx, 0) + \int \beta_1(1 - c_1 x) \mu(dx, 1), \\ \Lambda_x &= \int \alpha_0(1 - b_0 y) \hat{\mu}(dy, 0) + \int \alpha_1(1 - b_1 y) \hat{\mu}(dy, 1), \end{aligned}$$

where  $\mu$  is the invariant probability measure of  $(X_t, I_t)$  associated to equation:

$$X'_t = \alpha_{I_t} X_t (1 - a_{I_t} X_t),$$

and  $\hat{\mu}$  is the invariant probability measure of  $(Y_t, I_t)$  associated to equation:

$$Y'_t = \beta_{I_t} Y_t (1 - d_{I_t} Y_t).$$

The meaning of  $\Lambda_y$  is the following: when species  $y$  is close to extinction, species  $x$  behaves approximately as  $(X'_t, 0) = F_{\varepsilon_{I_t}}(X_t, 0)$  and  $\Lambda_y$  is the growth rate of species  $y$  with respect to invariant measure  $\mu$  of  $(X, I)$ . Note that the invasion rates depend on the jump rates  $(\lambda_0, \lambda_1) \in (0, +\infty)^2$ . For every  $(\lambda_0, \lambda_1) \in (0, +\infty)^2$ , we have two parametrizations of these jump rates:

$$(s, t) \in [0, 1] \times (0, +\infty) : \quad st = \lambda_0, \quad (1 - s)t = \lambda_1.$$

$$(u, v) \in [0, 1] \times (0, +\infty) : \quad uv = \lambda_0/\alpha_0, \quad (1 - u)v = \lambda_1/\alpha_1.$$

The change of parameters  $(u, v) = \xi(s, t)$  is triangular in the sense that  $u$  only depends on  $s$

$$(u, v) = \xi(s, t) = \left( \frac{s\alpha_1}{(1 - s)\alpha_0 + s\alpha_1}, \frac{t}{\alpha_0\alpha_1}((1 - s)\alpha_0 + s\alpha_1) \right).$$

Let us denote the invasion rates in the  $(u, v)$  coordinates by

$$\tilde{\Lambda}_x(u, v) = \Lambda_x(\xi^{-1}(u, v)) \quad \text{and} \quad \tilde{\Lambda}_y(u, v) = \Lambda_y(\xi^{-1}(u, v)).$$

It is established in [3] that signs of  $\tilde{\Lambda}_x$  and  $\tilde{\Lambda}_y$  determine the long time behavior of  $(X_t, Y_t)$ .

	$\tilde{\Lambda}_y > 0$	$\tilde{\Lambda}_y < 0$
$\tilde{\Lambda}_x > 0$	persistence of the two species	extinction of species $y$
$\tilde{\Lambda}_x < 0$	extinction of species $x$	extinction of species $x$ or $y$

Moreover, in [3] it is shown that two environments of Type 1 may lead to four regimes for the stochastic process. This surprising result is reminiscent of switched stable linear ODE studied in [1, 2].

A fundamental property of the model is that, for all  $0 \leq s \leq 1$ , the vector field  $(1-s)F_{\varepsilon_0} + sF_{\varepsilon_1}$  is the Lotka-Volterra system associated to the environment  $\varepsilon_s = (a_s, b_s, c_s, d_s, \alpha_s, \beta_s)$  with

$$\alpha_s = s\alpha_1 + (1-s)\alpha_0, \quad a_s = \frac{s\alpha_1 a_1 + (1-s)\alpha_0 a_0}{\alpha_s}, \quad b_s = \frac{s\alpha_1 b_1 + (1-s)\alpha_0 b_0}{\alpha_s}, \quad (1.1)$$

$$\beta_s = s\beta_1 + (1-s)\beta_0, \quad c_s = \frac{s\beta_1 c_1 + (1-s)\beta_0 c_0}{\beta_s}, \quad d_s = \frac{s\beta_1 d_1 + (1-s)\beta_0 d_0}{\beta_s}. \quad (1.2)$$

Set

$$I = \{0 \leq s \leq 1 : a_s > c_s\} \quad \text{and} \quad J = \{0 \leq s \leq 1 : b_s > d_s\}.$$

We denote by  $\tilde{I}$  the image of  $I$  for the other parametrization.

**Remark 1.1.** As noticed in [3], if  $\varepsilon_0$  and  $\varepsilon_1$  are of Type 1 then  $I$  or  $J$  may generically be empty or an open interval whose closure is contained in  $(0, 1)$ .

Let us recall below the key result in [6] about the expression of the invasion rates.

**Lemma 1.2.** [6, Lemma 1.2] Assume that  $\varepsilon_0$  and  $\varepsilon_1$  are of Type 1 and, w.l.g.,  $a_0 < a_1$ . The quantity  $\tilde{\Lambda}_y$  can be rewritten as:

$$\tilde{\Lambda}_y(u, v) = \frac{1}{(a_1 - a_0)\left(\frac{1}{\alpha_0}(1-u) + \frac{1}{\alpha_1}u\right)} \mathbb{E}[\phi(U_{u,v})]$$

where  $\phi : [0, 1] \rightarrow \mathbb{R}$  is defined by

$$\phi(y) = (a_0 + (a_1 - a_0)y)P\left(\frac{1}{a_0 + (a_1 - a_0)y}\right),$$

where

$$P(x) = \left(\frac{\beta_1}{\alpha_1}(1 - c_1x)(1 - a_0x) - \frac{\beta_0}{\alpha_0}(1 - c_0x)(1 - a_1x)\right) \frac{a_1 - a_0}{|a_1 - a_0|}, \quad (1.3)$$

and  $U_{u,v}$  is a Beta distributed  $\text{Beta}(uv, (1-u)v)$  random variable. Moreover,  $\phi$  has the following properties:

- If  $I$  is empty then  $\phi$  is nonpositive.
- If  $I$  is nonempty ( $I = (u_1, u_2)$ ) then  $\phi$  is concave, negative on  $(0, u_1) \cup (u_2, 1)$  and positive on  $\tilde{I} = (u_1, u_2)$ .

Our first result is the precise study of the properties of  $\tilde{\Lambda}_x$  and  $\tilde{\Lambda}_y$  with two environments  $\varepsilon_0, \varepsilon_1$  that are respectively of Type 1 and Type 2. In particular, we describe the regions where  $\tilde{\Lambda}_x$  and  $\tilde{\Lambda}_y$  are positive.

**Theorem 1.3.** (Shape of the regions). Assume that  $\varepsilon_0$  and  $\varepsilon_1$  are respectively of Type 1 and Type 2. Then, there exists a function  $u \mapsto v_y(u)$  from  $(0, 1) \rightarrow [0, \infty]$ , such that  $\tilde{\Lambda}_y(u, v) < 0$  when  $v < v_y(u)$  and  $\tilde{\Lambda}_y(u, v) > 0$  when  $v > v_y(u)$ . Let  $a$  be the coefficient of second degree of polynomial  $P$  given by (1.3).

If  $a < 0$ , there exists  $0 < \alpha < \bar{\alpha} < 1$  such that  $v_y$  is infinite on  $[0, \alpha]$ , is decreasing and continuous on  $(\alpha, \bar{\alpha})$ , tends to  $+\infty$  at  $\alpha$ , tends to 0 at  $\bar{\alpha}$  and is equal to 0 on  $[\bar{\alpha}, 1]$ .

If  $a > 0$ , there exists  $0 < \bar{\alpha} < \alpha < 1$  such that  $v_y$  is equal to 0 on  $[0, \bar{\alpha}]$ , is increasing and continuous on  $(\bar{\alpha}, \alpha)$ , tends to 0 at  $\bar{\alpha}$ , tends to  $+\infty$  at  $\alpha$ , and is infinite on  $[\alpha, 1]$ .

Moreover,  $\alpha$  and  $\bar{\alpha}$  are explicit.

The second result is the following theorem.

**Theorem 1.4.** *For any  $(i, j)$  in  $\{1, 2, 3, 4\}^2$ , there exist two environments  $\varepsilon_0$  of Type  $i$  and  $\varepsilon_1$  of Type  $j$  such that the associated stochastic process has four possible regimes depending on the jump rates.*

The paper is organized as follows. In Section 2 we prove the properties of  $\tilde{\Lambda}_x$  and  $\tilde{\Lambda}_y$ . In Section 3 we prove Theorem 1.3. In Section 4 we present illustrations obtained by numerical simulation. In Section 5 we study the case when the two environments are of Type 3. Finally, in Section 6, we prove Theorem 1.4 providing, in each case, a good couple of environments.

## 2 Expression of invasion rates

**Lemma 2.1.** *If  $\varepsilon_0$  and  $\varepsilon_1$  are respectively of Type 1 and Type 2, then  $\tilde{I}$  is always nonempty and there exists  $0 < \alpha < 1$  (depends on  $\alpha_i, \beta_i, a_i, c_i$ ) such that  $\tilde{I} = (\alpha, 1]$ .*

*Proof.* Set

$$R = \frac{\beta_0 \alpha_1}{\alpha_0 \beta_1}, \quad u = \frac{s \alpha_1}{\alpha_s}, \quad A = (a_1 - a_0)(R - 1), \quad B = (2a_0 - c_0 - a_1)R + (c_1 - a_0), \quad C = (c_0 - a_0)R.$$

For any  $s \in (0, 1)$ , we get that

$$c_s - a_s = \frac{Au^2 + Bu + C}{R(1 - u) + u}$$

where  $a_s$  and  $c_s$  are given by (1.1) and (1.2). Set

$$T(u) = Au^2 + Bu + C \quad \forall u \in [0, 1].$$

We easily get

$$T(0) = C = (c_0 - a_0)R > 0, \quad T(1) = A + B + C = c_1 - a_1 < 0.$$

Because  $T$  is a second degree polynomial with  $T(0) > 0$  and  $T(1) < 0$ , we conclude that

$$T(u) < 0 \Leftrightarrow u > \alpha = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

Therefore  $u \in \tilde{I} \Leftrightarrow T(u) < 0 \Leftrightarrow u > \alpha \Leftrightarrow u \in (\alpha, 1]$ . As a consequence,  $\tilde{I} = (\alpha, 1]$ . □

**Proposition 2.2.** *The map  $\tilde{\Lambda}_y(u, v)$  satisfies the following properties:*

*For all  $u \in [0, 1]$*

$$\lim_{v \rightarrow \infty} \tilde{\Lambda}_y(u, v) = \beta_u \left(1 - \frac{c_u}{a_u}\right) \begin{cases} > 0 & \text{if } u \in \tilde{I} = (\alpha, 1], \\ = 0 & \text{if } u \in \partial \tilde{I} = \{\alpha\}, \\ < 0 & \text{if } u \in (0, 1) \setminus \tilde{I} = [0, \alpha), \end{cases}$$

*and*

$$\lim_{v \rightarrow 0} \tilde{\Lambda}_y(u, v) = \frac{1}{\frac{1}{\alpha_0}(1 - u) + \frac{1}{\alpha_1}u} \left( \left( \frac{\beta_1}{\alpha_1} \left(1 - \frac{c_1}{a_1}\right) - \frac{\beta_0}{\alpha_0} \left(1 - \frac{c_0}{a_0}\right) \right) u + \frac{\beta_0}{\alpha_0} \left(1 - \frac{c_0}{a_0}\right) \right). \quad (2.1)$$

*Proof.* The proposition is obtained by changing variables  $(s, t) \longleftrightarrow (u, v)$  from [3, Prop. 2.3].  $\square$

**Proposition 2.3.** *There exists  $0 < \bar{\alpha} < 1$  such that  $\lim_{v \rightarrow 0} \tilde{\Lambda}_y(u, v) > 0$  if  $u > \bar{\alpha}$  and  $\lim_{v \rightarrow 0} \tilde{\Lambda}_y(u, v) < 0$  if  $u < \bar{\alpha}$ .*

*Proof.* The limit in (2.1) has the same sign than

$$g(u) = \left( \frac{\beta_1}{\alpha_1} \left(1 - \frac{c_1}{a_1}\right) - \frac{\beta_0}{\alpha_0} \left(1 - \frac{c_0}{a_0}\right) \right) u + \frac{\beta_0}{\alpha_0} \left(1 - \frac{c_0}{a_0}\right), \quad \forall u \in [0, 1].$$

We get

$$g(0) = \frac{\beta_0}{\alpha_0} \left(1 - \frac{c_0}{a_0}\right) < 0 \quad \text{and} \quad g(1) = \frac{\beta_1}{\alpha_1} \left(1 - \frac{c_1}{a_1}\right) > 0.$$

Since  $g$  is a linear function,  $\bar{\alpha}$  is the unique zero of  $g$  and the result is clear.  $\square$

**Proposition 2.4.** *Let  $a$  be the coefficient of degree 2 of polynomial  $P$  given by (1.3)*

$$a = \left( \frac{\beta_1}{\alpha_1} c_1 a_0 - \frac{\beta_0}{\alpha_0} c_0 a_1 \right) \frac{a_1 - a_0}{|a_1 - a_0|}.$$

*If  $a < 0$  (resp.  $a > 0$  or  $a = 0$ ) then  $\alpha < \bar{\alpha}$  (resp.  $\alpha > \bar{\alpha}$  or  $\alpha = \bar{\alpha}$ ).*

*Proof.* By symmetry we only consider the case  $a < 0$ . Without loss of generality, we assume that  $a_1 > a_0$  and  $a$  becomes:

$$a = \frac{\beta_1}{\alpha_1} c_1 a_0 - \frac{\beta_0}{\alpha_0} c_0 a_1.$$

To prove that  $\alpha < \bar{\alpha}$ , it is sufficient to prove  $A\bar{\alpha}^2 + B\bar{\alpha} + C < 0$ . Since, by definition of  $\bar{\alpha}$ ,

$$\left( \frac{\beta_1}{\alpha_1} \left(1 - \frac{c_1}{a_1}\right) - \frac{\beta_0}{\alpha_0} \left(1 - \frac{c_0}{a_0}\right) \right) \bar{\alpha} + \frac{\beta_0}{\alpha_0} \left(1 - \frac{c_0}{a_0}\right) = 0,$$

we get, multiplying by  $a_0 a_1 \alpha_1 / \beta_1$ , that

$$(a_0 a_1 - c_1 a_0 - R a_1 a_0 + R a_1 c_0) \bar{\alpha} + R a_1 (a_0 - c_0) = 0. \quad (2.2)$$

Replacing  $\bar{\alpha}$  by its expression in (2.2), we get:

$$A\bar{\alpha}^2 + B\bar{\alpha} + C = \frac{R(a_1 - c_1)(a_0 - a_1)(a_0 - c_0)(a_0 c_1 - R a_1 c_0)}{(a_0 a_1 - a_0 c_1 - R a_0 a_1 + R a_1 c_0)^2}.$$

Since  $c_0 > a_0, a_1 > c_1, a_1 > a_0$  and  $a_0 c_1 - R a_1 c_0 = \frac{\alpha_1}{\beta_1} a < 0$ , we conclude  $A\bar{\alpha}^2 + B\bar{\alpha} + C < 0$ .  $\square$

### 3 Shape of the positivity region

Recall  $a = \left( \frac{\beta_1}{\alpha_1} a_0 c_1 - \frac{\beta_0}{\alpha_0} a_1 c_0 \right) \frac{a_1 - a_0}{|a_1 - a_0|}$  is the coefficient of degree 2 of polynomial  $P$  given by (1.3).

**Lemma 3.1.** [6, Lemma 4.1] *Assume  $\varepsilon_0$  and  $\varepsilon_1$  are of Type 1. If  $\tilde{I}$  is nonempty, then the map  $(u, v) \rightarrow \mathbb{E}[\phi(U_{u,v})]$  is increasing in  $v$  and concave in  $u$ .*

**Remark 3.2.** In Benaiïm and Lobry's case, if  $I$  is nonempty,  $\phi$  is concave and the parameter  $a$  is always negative. In the present case,  $a$  may be negative, positive or zero. Therefore, we have the following lemma.

**Lemma 3.3.** Assume  $\varepsilon_0$  and  $\varepsilon_1$  are respectively of Type 1 and Type 2, then the shape of  $\phi$  depends on the sign of  $a$ :

- If  $a$  is negative, then  $\phi$  is strongly concave and  $(u, v) \rightarrow \mathbb{E}[\phi(U_{u,v})]$  is increasing in  $v$  and concave in  $u$ .
- If  $a$  is positive, then  $\phi$  is strongly convex and  $(u, v) \rightarrow \mathbb{E}[\phi(U_{u,v})]$  is decreasing in  $v$  and convex in  $u$ .
- If  $a$  is zero, then  $\phi$  is linear and  $(u, v) \rightarrow \mathbb{E}[\phi(U_{u,v})]$  is constant in  $v$  and linear in  $u$ .

*Proof.* This is a straightforward adaptation of [6, Lem 4.1].  $\square$

Let us conclude this section with the proof of Theorem 1.3.

*Proof of Theorem 1.3.* We consider only the case  $a < 0$ . Set  $K = (\alpha, \bar{\alpha})$ . We know clearly that  $v \rightarrow \tilde{\Lambda}_y(u, v)$  admits:

- negative limits at 0 and  $\infty$  if  $u \in [0, \alpha)$ ,
- positive limits at 0 and  $\infty$  if  $u \in (\bar{\alpha}, 1]$ ,
- a negative limit at 0 and a positive limit at  $\infty$  if  $u \in (\alpha, \bar{\alpha})$ .

The fact that  $v \mapsto \tilde{\Lambda}_y(u, v)$  is increasing justifies the existence of  $v_y$ , and we have

$$\tilde{\Lambda}_y(u, v) = 0 \Leftrightarrow u \in K, v = v_y(u).$$

Let us prove that  $v_y$  is decreasing in  $K$ . Let  $\delta_1 < \delta_2$  be two points in  $K$ . Choose any  $\delta_3 \in (\bar{\alpha}, 1)$ , we get  $\tilde{\Lambda}_y(\delta_1, v_y(\delta_1)) = 0$  and  $\tilde{\Lambda}_y(\delta_3, v_y(\delta_1)) > 0$ . Since  $\tilde{\Lambda}_x(\cdot, v_y(\delta_1))$  is concave and  $\delta_1 < \delta_2 < \delta_3$  we get  $\tilde{\Lambda}_y(\delta_2, v_y(\delta_1)) > 0$ . Since  $\tilde{\Lambda}_y(\delta_2, \cdot)$  is increasing, we obtain  $v_y(\delta_2) < v_y(\delta_1)$ .

The continuity of  $v_y$  on  $K$  is a straightforward consequence of the continuity of the function  $\tilde{\Lambda}_y$ , which is obvious from the expression (1.2).

Let us show  $v_y$  tends to  $\infty$  on  $\alpha$ . Let  $\{u_n\} \subset K : u_n \downarrow \alpha$ . Since  $v_y$  is decreasing in  $K$ , we get  $v_y(u_n) \uparrow v \in [0, \infty]$ . If  $v$  is finite, since the zero set of  $\tilde{\Lambda}_y$  is closed, by continuity,  $\alpha \in K$  (impossible). So  $v_y(u_n) \uparrow \infty$ .

Let us prove  $v_y$  tends to 0 on  $\bar{\alpha}$ . Let  $\{u_n\} \subset K : u_n \uparrow \bar{\alpha}$ . Since  $v_y$  is decreasing in  $K$ , we get  $v_y(u_n) \downarrow \epsilon \in [0, \infty)$ . If  $\epsilon > 0$ , since  $u_n < \bar{\alpha}$ , we obtain  $\tilde{\Lambda}_y(u_n, \epsilon/2) < 0 \forall n$ . Therefore  $0 < \tilde{\Lambda}_y(\bar{\alpha}, \epsilon/2) = \lim_{n \rightarrow \infty} \tilde{\Lambda}_y(u_n, \epsilon/2) \leq 0$  (impossible). As a consequence,  $\epsilon = 0$  and  $v_y(u_n) \downarrow 0$ .  $\square$

## 4 Numerical illustrations

Recall that for all  $u \in [0, 1]$ ,  $v_y(u)$  and  $v_x(u)$  are the unique respective solutions of

$$\tilde{\Lambda}_y(u, v) = 0 \quad \text{and} \quad \tilde{\Lambda}_x(u, v) = 0.$$

We now consider, for a varying parameter  $\rho$ , the environments

$$\varepsilon_0 = (1, 5, 2, 8, 3, 3) \quad \text{and} \quad \varepsilon_1 = (2, 11, 1, \rho, 2, 1.8). \quad (4.1)$$

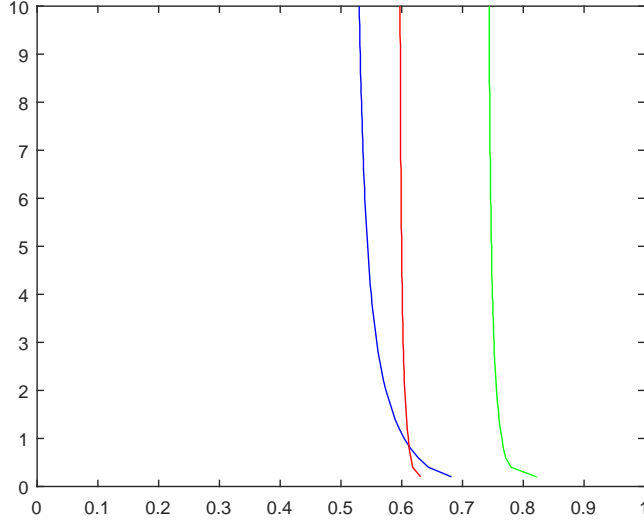


Figure 1: The blue curve is the graph of  $v_y$  (it does not depend on  $\rho$ ); the green and red curves are  $v_x$  for the environments given in (4.1) with  $\rho = 10$  and  $\rho = 9$  respectively.

Figure 1 represents the "critical" functions  $v_y$  and  $v_x$  for different choices of the environments. Thanks to [3], these plots give us information about how many regimes we can observe when the jump rates are modified. For example, the plot for  $\rho = 10$  has three regimes: extinction of  $x$  (on the right of the green curve), persistence (between the green and blue curves) and extinction of  $y$  (on the left of the blue curve). For  $\rho = 9$ , there is an additional zone (above the red curve and below the blue curve) that corresponds to jump rates leading to random extinction of a species.

## 5 Switching between two persistent Lotka-Volterra systems

Let us assume that  $\varepsilon_0$  and  $\varepsilon_1$  are of Type 3. In this case, one can easily get that extinction of species  $y$  is not possible if  $u$  is too close to 0 or 1; in other words,  $[0, 1] \setminus \tilde{I}$  is either empty or is an open interval whose closure is contained in  $[0, 1]$ . Recall

$$R = \frac{\beta_0 \alpha_1}{\alpha_0 \beta_1}, \quad A = (a_1 - a_0)(R - 1), \quad B = (2a_0 - c_0 - a_1)R + (c_1 - a_0), \quad C = (c_0 - a_0)R.$$

Then, we get that

$$[0, 1] \setminus \tilde{I} \neq \emptyset \Leftrightarrow \begin{cases} A < 0 \\ \Delta = B^2 - 4AC > 0 \\ 0 < \frac{-B - \sqrt{\Delta}}{2A} < 1. \end{cases}$$

Moreover, if  $[0, 1] \setminus \tilde{I}$  is nonempty, then the map  $(u, v) \rightarrow \mathbb{E}[\phi(U_{u,v})]$  is (strictly) decreasing in  $v$  and convex in  $u$ . This is a straightforward adaptation of Lemma 4.1 in [6].

Figure 2 provides the shape of  $v_x$  and  $v_y$  for the environments  $\varepsilon_0 = (6, 1, 4, 2, 1, 5)$  and  $\varepsilon_1 = (3, 3, 2, 5.5, 5, 1)$ . Once again, the switched process has four regimes depending on the jump rates.

**Remark 5.1.** *We see a surprising result : although both vector fields are persistent, the stochastic process may lead to the extinction of one of the two species.*

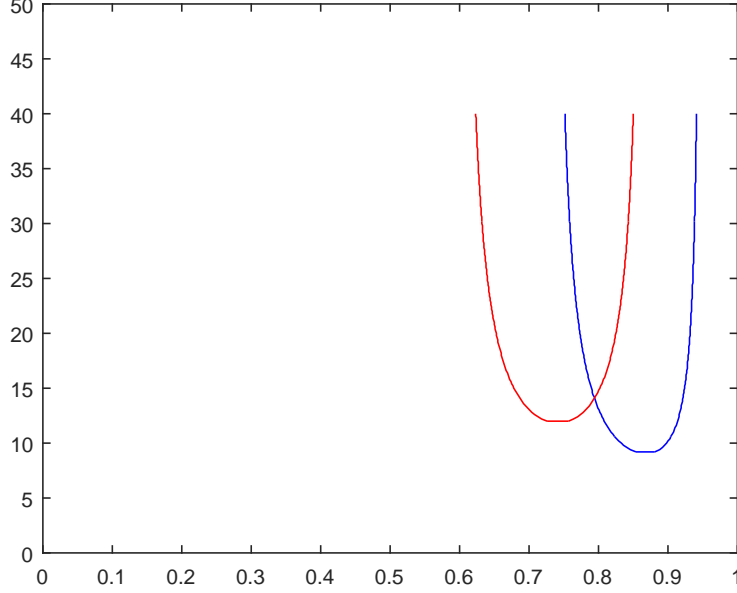


Figure 2: Graph of  $v_y$  (blue curve) and  $v_x$  (red curve) for the environments  $\varepsilon_0 = (6, 1, 4, 2, 1, 5)$  and  $\varepsilon_1 = (3, 3, 2, 5.5, 5, 1)$ .

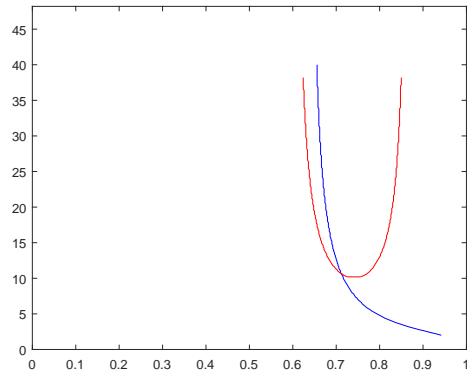
## 6 General case: proof of Theorem 1.4

The following array presents, for any couple of types, an example of two environments that are associated to a stochastic process with four regimes depending on the jump rates. The first line has been obtained in [3]. The second line is studied in Section 2. The fifth line is studied in Section 5. The reader can easily check that the other cases correspond to Figure 3.

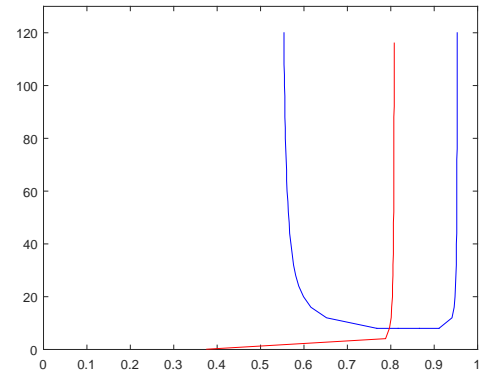
$(F_0, F_1)$	$a_0$	$b_0$	$c_0$	$d_0$	$\alpha_0$	$\beta_0$	$a_1$	$b_1$	$c_1$	$d_1$	$\alpha_1$	$\beta_1$
Type 1-1	1	1	2	2	1	5	3	3	4	3.5	5	1
Type 1-2	1	5	2	8	3	3	2	11	1	9	2	1.8
Type 1-3	1	1	3.5	2	1	5	5	3	4	5.5	5	1
Type 1-4	1	1	2	3.5	1	5	3	4	4	3	5	1
Type 3-3	6	1	4	2	1	5	3	3	2	5.5	5	1
Type 3-4	6	1	4	8	1	5	3	10	4	7	5	1
Type 4-4	2	2	1	1	5	1	7	3.5	4	3	1	5

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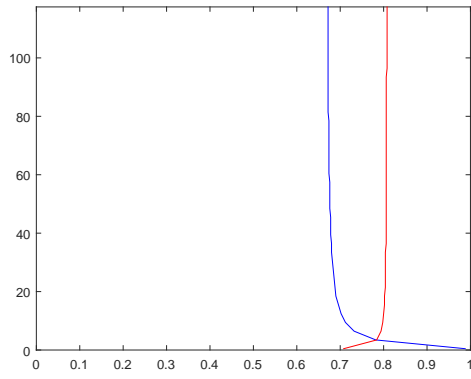




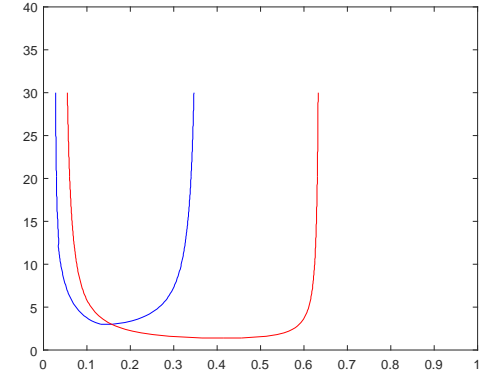
(a) Type 1-3



(b) Type 1-4



(c) Type 3-4



(d) Type 4-4

Figure 3: Graph of  $v_y$  (blue curve) and  $v_x$  (red curve) for the four last cases.

## References

- [1] Y. Bakhtin and T. Hurth, *Invariant densities for dynamical systems with random switching*, Nonlinearity **25** (2012), no. 10, 2937–2952.
- [2] M. Benaïm, S. Le Borgne, F. Malrieu, and P.-A. Zitt, *On the stability of planar randomly switched systems*, Ann. Appl. Probab. **24** (2014), no. 1, 292–311. MR 3161648
- [3] M. Benaïm and C. Lobry, *Lotka Volterra with randomly fluctuating environments or "how switching between beneficial environments can make survival harder"*, Preprint available on arXiv number 1412.1107. To appear in Annals of Applied Probability, 2015.
- [4] M. H. A. Davis, *Piecewise-deterministic Markov processes: a general class of nondiffusion stochastic models*, J. Roy. Statist. Soc. Ser. B **46** (1984), no. 3, 353–388, With discussion. MR 790622
- [5] F. Malrieu, *Some simple but challenging Markov processes*, Ann. Fac. Sci. Toulouse Math. (6) **24** (2015), no. 4, 857–883. MR 3434260
- [6] F. Malrieu and P.-A. Zitt, *On the persistence regime for Lotka-Volterra in randomly fluctuating environments*, Preprint available on arXiv number 1601.08151, 2016.

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